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GENERALIZED BEAM THEORY AND MODULAR STRUCTURES

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Abstract—A modular structure is one which is formed from a repeated module, most commonly a regular truss. These are often reduced to equivalent uniform continua so that standard beam theory can be applied. This paper examines the basis of standard beam theory, demonstrating it to be a particular case of a more general theory. It is shown that more accurate results can be obtained by a proper application of the theory to both discrete modular systems and to normal continuous beams.

1. THE CHARACTERISTIC RESPONSE

The general theory of beams was discussed in an earlier paper (Renton, 1991). It is not based on the Bernoulli–Euler theory but on Saint-Venant's principle and the principle of superposition for small deflections of linearly-elastic materials.

Consider the infinitely long modular beam shown in Fig. 1(a). A resultant moment M is applied to the left-hand side of the 0th module (the resultant shear force S is zero in this instance). The Xth module is close to it and the Nth module is a long distance from it. The resultant moment on the Xth module will also be M. Suppose that the structure is sectioned just to the left of the Xth module, but that the stress interactions are still applied to it. This can be compared with the original beam shifted X modules to the right. From the principle of superposition, an admissible stress pattern is given by taking the difference of the patterns in the two cases. This has a zero resultant moment at the left-hand end, and applying Saint-Venant's principle, the stress and strain pattern in the $(N–X)$th module must be sensibly zero too. This means that in the original beam, although the patterns of stress and strain may not be uniform within a module, the patterns in the $(N–X)$th and the Nth modules are sensibly the same. This leads to Theorem 1:

> If only a resultant moment is applied to the end of a modular beam, at large distances along it, the patterns of stress and strain in each module tend towards a unique response, which will be called the *characteristic response to a moment*. Conversely, if an admissible stress pattern can be found, which is the same in all modules and only has a resultant moment, this must be the characteristic response

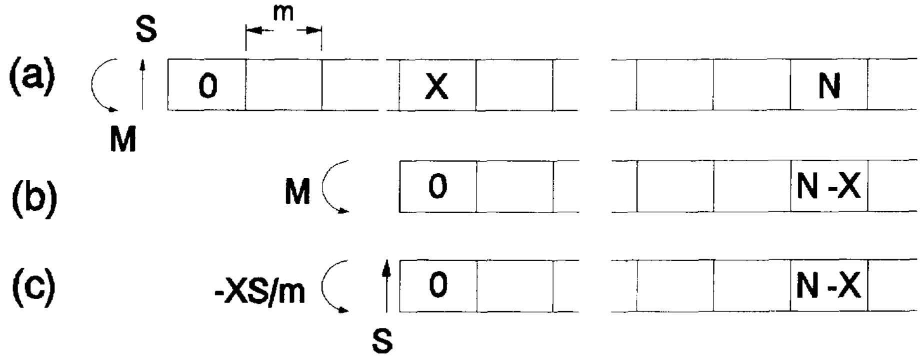

Fig. 1. Response of a modular beam.

to a moment. If this were not so, the difference between the two responses would violate Saint-Venant's principle.

For Saint-Venant's principle to apply, the individual modules must not be mechanisms. An example of this exception is given by Hoff (1945).

Returning to Fig. 1(a), consider the case when only the shear force S is applied (M is zero). If the length of each module is m, then the resultants on the left-hand face of the Xth module will the shear force S and a bending moment $-XS/m$. By the same arguments as before, the beam can be sectioned here and the difference taken between the response of the beam to the right and the response of the shifted beam, to which a moment $-XS/m$ has been added. This should again produce stress and strains in the (N–X)th module which are sensibly zero. This means that the difference in response between that in the original Nth module and that in the original (N–X)th module is given by the characteristic response to a moment of $-XS/m$. This leads to Theorem 2:

> If a shear force is applied to the end of a modular beam, the patterns of stress and strain in modules far from the end can be divided into a linearly-varying component which is the characteristic response to a moment and a constant component called the *characteristic response to a shear force*. By the same arguments as before, if a pattern of stress can be found which varies linearly between modules and has only a resultant shear force and bending moment, this must be a combination of the characteristic responses to each.

Beam theory relies on the fact that any loading produces stresses and strains which tend to decay towards the characteristic responses along the beam, and these responses alone can represent the loaded state with sufficient accuracy for most engineering purposes, provided that Saint-Venant's principle applies strongly enough. Similar arguments to the above apply to ordinary beams in which the modules are replaced by a prismatic continuum. No assumption is made that plane sections remain plane. The constant (characteristic) response of an anisotropic beam to a bending moment is given by Lekhnitskii (1981). For a bending moment M_y about the principal axis y of a beam with x as its centroidal axis, the displacements in the x,y and z directions, u,v and w, can be written as

$$u = \frac{M_y}{2I}(2s_{11}xz+s_{16}yz+s_{15}z^2)$$
$$v = \frac{M_y}{2I}(s_{16}xz+2s_{12}yz+s_{14}z^2)$$
$$w = \frac{M_y}{2I}(-s_{16}xy-s_{11}x^2-s_{12}y^2+s_{13}z^2). \quad (1)$$

Here I is the second moment of area about the y axis and the coefficents s_{ij} are compliances of the material. If it is isotropic, they take the values

$$s_{11} = \frac{1}{E}; \quad s_{12} = s_{13} = -\frac{v}{E}; \quad s_{14} = s_{15} = s_{16} = 0, \quad (2)$$

where E is Young's modulus and v is Poisson's ratio. The stresses and strains are given by

$$\sigma_{xx} = \frac{M_y z}{I}; \quad \sigma_{yy} = \sigma_{zz} = \tau_{yz} = \tau_{zx} = \tau_{xy} = 0,$$
$$\frac{\varepsilon_{xx}}{s_{11}} = \frac{\varepsilon_{yy}}{s_{12}} = \frac{\varepsilon_{zz}}{s_{13}} = \frac{\gamma_{yz}}{s_{14}} = \frac{\gamma_{zx}}{s_{15}} = \frac{\gamma_{xy}}{s_{16}} = \frac{M_y z}{I}. \quad (3)$$

For an isotropic beam, these expressions become the usual ones associated with the Bernoulli–Euler theory. Plane sections remain plane because for a particular value of x, u

varies linearly with z. In general however, the characteristic response produces a curved cross-section. One result of this, as will be seen later, is that it is necessary to rethink what is meant by the rotation of a particular cross-section.

2. COUPLING EFFECTS

If a prismatic beam is in a constant state of strain along its axis, the most general expressions for the displacements can be written as

$$u = x(\varepsilon - \psi_z y + \psi_y z) + U(y,z)$$
$$v = -xz\theta + \tfrac{1}{2}\psi_z x^2 + V(y,z)$$
$$w = xy\theta - \tfrac{1}{2}\psi_y x^2 + W(y,z) \tag{4}$$

excluding arbitrary rigid-body motions. Here, θ is a uniform rate of twist, ψ_y and ψ_z are uniform rates of curvature about the y and z axes and ε is a uniform axial strain. Comparing these expressions with eqn (1) shows that

$$\psi_y = \frac{M_y}{I}s_{11}, \quad \theta = -\frac{M_y}{2I}s_{16} \tag{5}$$

so that a bending moment can produce torsion in an anisotropic beam. The most general relationship between the load resultants and the corresponding deformations of a beam can be written as

$$\begin{bmatrix} \theta \\ \psi_y \\ \psi_z \\ \varepsilon \\ \gamma_y \\ \gamma_z \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ f_{31} & f_{32} & f_{33} & f_{34} & f_{35} & f_{36} \\ f_{41} & f_{42} & f_{43} & f_{44} & f_{45} & f_{46} \\ f_{51} & f_{52} & f_{53} & f_{54} & f_{55} & f_{56} \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix} \begin{bmatrix} T \\ M_y \\ M_z \\ P \\ S_y \\ S_z \end{bmatrix} \tag{6}$$

where γ_y and γ_z are the shear strains corresponding to the resultant shear forces S_y and S_z in the y and z directions, T is the resultant torque, P is the resultant axial force and it follows from Betti's reciprocal theorem that the matrix of flexibility coefficients f_{ij} is symmetric.

A degree of uncoupling can be achieved by defining the coordinates (y_o, z_o) of the line of action of P to be such that it does not produce bending, thus eliminating f_{24} and f_{34} (and by implication f_{42} and f_{43}). This would normally be referred to as the 'centroid' of the section, although more generally it has no geometric significance. Likewise, it is possible to define a 'shear centre' eliminating the coupling between torsion and shear given by the coefficients f_{15} and f_{16} and 'principal axes' y and z eliminating the flexural coupling given by f_{23}.

The shear flexibility is determined from the shear strain energy per unit length, U_s, stored in the characteristic response to a shear force. This has components related to the linearly-varying bending moment as well as those directly attributable to shear. If x is the distance along the beam from a section at which the resultant load is a shear force S, it takes the form

$$U_s = \tfrac{1}{2}f_{mm}(Sx)^2 + f_{ms}S^2x + \tfrac{1}{2}f_{ss}S^2 \tag{7}$$

where f_{mm}, f_{ms} and f_{ss} are f_{22}, f_{26} and f_{66} for bending about the y axis or f_{33}, f_{35} and f_{55} for bending about the z axis. For ordinary beams, no coupling terms f_{ms} have been found.

However, for modular beams, these terms do exist. This is because the module is of finite length so that there is a degree of arbitrariness about the definition of the bending moment acting on it in the presence of a shear force. By analogy with the above definitions of 'centroid' and 'shear centre', the bending moment will be defined as that acting at a point along the module such that no coupling between bending and shear occurs, so eliminating f_{ms}. This point will be referred to as the *flexural centre* of the module. This is normally, but not always, at the midsection of the module.

3. END CONDITIONS

It has been argued that the end conditions for ordinary beams are well understood and that they only apply weakly to modular structures such as trusses. As will be seen, this is not necessarily true. Consider the plane-stress analysis of a cantilever given by Timoshenko and Goodier (1970) and others, Dugdale and Ruiz (1971) and Ford and Alexander (1977) for example. It is of unit thickness, length l, depth $2c$, and a parabolic shear stress distribution is applied to the free end with a downwards resultant denoted here by F. All the internal and boundary conditions are satisfied except those at the fixed end. The x axis lies along the middle line of the beam, but the origin will be taken at the fixed end, for reasons which will become apparent later. The expressions for the deflections then take the form

$$u = \frac{Fy}{2EI}(l-x)^2 - \frac{Fy^3}{6EI}(2+\nu) - ey + g$$

$$v = \frac{\nu F y^2}{2EI}(l-x) + \frac{F}{6EI}(l-x)^3 + d(l-x) + h \tag{8}$$

where

$$d+e = -\frac{Fc^2}{2GI}, \quad I = \tfrac{2}{3}c^3$$

and G is the shear modulus.

The constants d to h are determined from the conditions that the centroid does not move at the fixed end, and locally the vertical face at the centroidal axis does not rotate. This implies that the fixed-end horizontal displacement is

$$u_0 = -\frac{Fy^3}{6EI}(2+\nu) \tag{9}$$

as shown in Fig. 2(a), and the vertical displacement of the middle line is given by

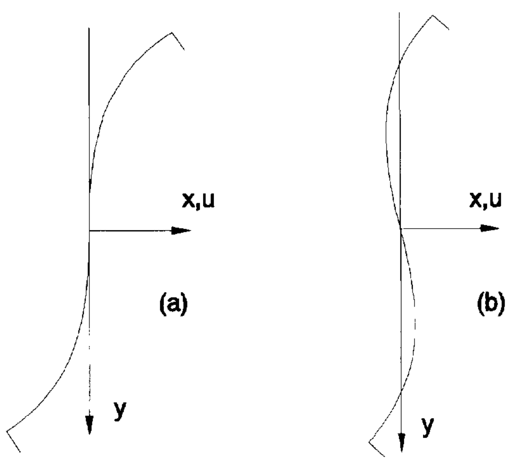

Fig. 2. Horizontal displacements at a fixed end.

$$v_c = \frac{F}{EI}[\tfrac{1}{6}(-x^3+3x^2l)+c^2x(1+\nu)]. \tag{10}$$

However, this gives undue weight to the conditions in the immediate vicinity of the origin.

The solution given by eqn (8) has the properties of the characteristic response to bending and shear described above. The associated beam theory is a macroscopic analysis relating resultant moments and shear forces to the corresponding rotations and displacements through which they do work. These rotations and displacements may have no relationship to those of a particular zone on the cross-section. Taking the fixed end to provide a workless reaction, the appropriate conditions to use on the end section are

$$\int_{-c}^{c} \sigma_{xx} u \, dy = \int_{-c}^{c} \tau_{xy} v \, dy = 0 \tag{11}$$

giving in particular

$$u_0 = \frac{Fy}{EI}(2+\nu)\left(\frac{c^2}{10}-\frac{y^2}{6}\right) \tag{12}$$

as shown in Fig. 2(b) and

$$v_c = \frac{F}{EI}\left[\frac{1}{6}(-x^3+3x^2l)+\frac{c^2x}{10}(8+9\nu)-\frac{\nu c^2 l}{10}\right]. \tag{13}$$

By measuring x from the fixed end, only the cubic and quadratic terms in eqns (10) and (13) are associated with normal flexure, and are the same in both cases. The remaining terms are related to shear and rigid-body motion.

On the basis of the theory described above, a good match should not be expected in the vicinity of the ends, but local effects should decay along the beam towards the characteristic response. This may still leave an error, which is expressed by a rigid-body displacement and rotation.

This can be investigated experimentally, using finite-element analysis. Figure 3 shows the much-exaggerated deflected form of a cantilever 1 m long, 10 cm deep and 1 cm wide. It is divided into a thousand square plane-stress finite elements. A shear force of 1 kN is applied to the free end, using a parabolic distribution of shear forces on the elements. All displacements of the elements are prevented at the fixed end.

A best-fit cubic polynomial for v_c was generated from values at equally-spaced data points in the computer analysis. To avoid local effects, points within 10 cm (the depth of the beam) of the fixed end were not used. The cubic was interpolated from the results for different data samples, looking for the best match with the flexural terms. This was found to be

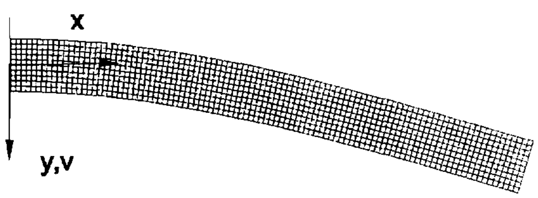

Fig. 3. Finite-element representation of a cantilever.

$$v_c = [-967.0x^3+2{,}901x^2+11.81x-0.565]\,\mu\text{m} \tag{14}$$

where x is in metres, Young's modulus is 206.8 GPa, and Poisson's ratio is 0.29. The corresponding result given by eqn (10) is

$$v_c = [-967.1x^3+2{,}901x^2+18.71x]\,\mu\text{m} \tag{15}$$

and that given by eqn (13) is

$$v_c = [-967.1x^3+2{,}901x^2+15.39x-0.421]\,\mu\text{m}. \tag{16}$$

It will be seen that the above expression for workless reactions corresponds more closely than eqn (15) to the finite-element analysis. This in turn indicates that truly fixed ends provide even more constraint against deflection.

4. THE GOVERNING EQUATIONS FOR MODULAR BEAMS

Suppose that the middle section of a typical module is at a distance x from an end where a pure shear force S is applied, and that the analysis of its characteristic response shows that the strain energy stored in the module is mU_s where U_s is given by eqn (7). The flexural centre is a further distance me along the module, and at this point the expression separates out into its flexural and shear components,

$$U_s = \tfrac{1}{2}f_{mm}[S(x+me)]^2+\tfrac{1}{2}f'_{ss}S^2 \tag{17}$$

where

$$me = f_{ms}/f_{mm}, \quad f'_{ss} = f_{ss}-f^2_{ms}/f_{mm}.$$

In order to aid comparisons with ordinary beam theory, these flexibilities will be denoted by the more familiar expressions

$$f_{mm} = \frac{1}{EI}, \quad f'_{ss} = \frac{1}{K_s} \tag{18}$$

although in general the form given by eqn (6) is preferable.

Beam problems would normally be cast and solved in terms of differential equations and integrations. For modular beams, this can be done in terms of difference equations and summations. The functions take integer parameters, and the operators used here are defined by

$$\begin{aligned} \mathrm{E}f(X) &= f(X+1) \\ \Delta f(X) &= f(X+1)-f(X) = (\mathrm{E}-1)f(X) \\ \nabla f(X) &= f(X)-f(X-1) = (1-\mathrm{E}^{-1})f(X) \\ \Delta\nabla f(X) &= \Delta[\nabla f(X)] = (\mathrm{E}+\mathrm{E}^{-1}-2)f(X). \end{aligned} \tag{19}$$

It is usually convenient to count the modules of a beam starting from zero, with the Nth being a virtual module whose left-hand side is the right-hand end of the beam, as shown in

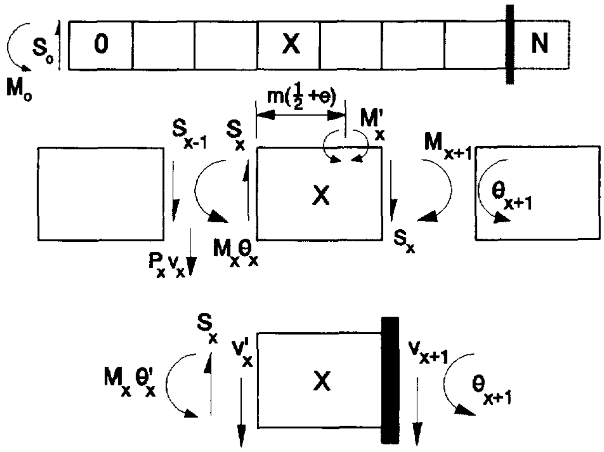

Fig. 4. Basic relationships for modular beams.

Fig. 4. Details of the Xth element are shown, the moments acting on its left- and right-hand faces being M_X and M_{X+1} respectively, in the senses shown. The shear force acting on the module is S_X and the moment at the flexural centre is M_X', (hogging positive). Distributed loads are applied at the module junctions, that acting at the left-hand junction of module X being P_X. The displacement and rotation through which P_X and M_X do work are v_X and θ_X respectively. For equilbrium,

$$\begin{aligned} P_X &= S_{X-1} - S_X = -\nabla S_X \\ S_X m &= M_X - M_{X+1} = -\Delta M_X \\ M'_X &= M_X - S_X m(\tfrac{1}{2}+e). \end{aligned} \tag{20}$$

The strain energy in the module is given by

$$mU = \frac{mM_X'^2}{2EI} + \frac{mS_X^2}{2K_s}. \tag{21}$$

Suppose for the moment that the right-hand side of this module is fixed, so that both v_{X+1} and θ_{X+1} are zero. Then from Castigliano's theorem, the deflections of the left-hand side are given by

$$v'_X = -m\frac{\partial U}{\partial S_X}, \quad \theta'_X = m\frac{\partial U}{\partial M_X}. \tag{22}$$

If a rigid-body motion is now given to the system so that the right-hand side of the module is displaced through v_{X+1} and rotated through θ_{X+1}, the general expressions for the displacement and rotation of the left-hand side of the module are found to be

$$v_X = v'_X + v_{X+1} + m\theta_{X+1}, \quad \theta_X = \theta'_X + \theta_{X+1}. \tag{23}$$

For convenience, v_X will be divided into its components, $v_X{}^s$ associated with the shear of the module and $v_X{}^m$ associated with the flexure of the module, so that from eqns (20) to (23),

$$\Delta v_X^s = \frac{mS_X}{K_s}$$

$$\Delta v_X^m = (\tfrac{1}{2}+e)\frac{m^2}{EI}[mS_X(\tfrac{1}{2}+e)-M_X]-m\theta_{X+1}$$

$$\Delta\theta_X = \frac{m}{EI}[mS_X(\tfrac{1}{2}+e)-M_X]. \tag{24}$$

These can either be solved directly, or further use made of eqn (20) to derive the forms

$$\Delta\nabla v_X^s = -\frac{mP_X}{K_s}$$

$$(\Delta\nabla)^2 v_X^m = \frac{m^3}{EI}[1+\Delta\nabla(\tfrac{1}{4}-e^2)]P_X$$

$$\Delta^2\nabla\theta_X = -\frac{m^2}{EI}[1+\Delta(\tfrac{1}{2}+e)]P_X. \tag{25}$$

Taking the ordinate x as equivalent to mX, the distributed load p_x as equivalent to P_X/m and using the Taylor expansion

$$\mathrm{E}f(x) = f(x+m) = f(x)+m\frac{\mathrm{d}f}{\mathrm{d}x}+\frac{m^2}{2}\frac{d^2f}{\mathrm{d}x^2}+\cdots$$

and then allowing m to tend to zero in the above equations, gives at the limit

$$\frac{\mathrm{d}^2v_x^s}{\mathrm{d}x^2} = -\frac{p_x}{K_s},\quad \frac{\mathrm{d}^4v_x^m}{\mathrm{d}x^4} = \frac{p_x}{EI},\quad \frac{\mathrm{d}^3\theta_x}{\mathrm{d}x^3} = -\frac{p_x}{EI} \tag{26}$$

which is the continuum equivalent of eqn (25). [Values of K_s for the continuum problem have been given previously, Renton (1991).]

5. SOME STANDARD SOLUTIONS

As in eqn (26), the shear displacement, v_x^s, of ordinary prismatic beams is governed by a second-order differential equation. The two constants of integration correspond to two boundary conditions, one at each end (relating to shear force and displacement). The flexural displacement, v_x^m, is governed by a fourth-order equation, implying two boundary conditions at each end (relating to bending moment, shear force, rotation and displacement). The corresponding boundary conditions for the modular beam are exactly the same.

As an example, suppose that the cantilever shown at the top of Fig.4 is loaded only by linearly-varying loads at the module junctions given by

$$P_X = XP.$$

The end conditions at the left- and right-hand ends are then

$$M_0 = S_0 = 0 \quad \text{and} \quad \theta_N = y_N = 0,$$

respectively. Formulae required in the following solutions will be found in the Appendix.

From statics,

$$S_X = -\sum_{I=1}^{X} IP = -\tfrac{1}{2}(X^2+X)P, \quad M_X = \sum_{I=1}^{X} m(X-I)IP = \frac{m}{6}(X^3-X)p$$

so that from eqn (24),

$$\Delta v_X^s = -\frac{mP}{2K_s}(X^2+X)$$

$$\Delta\theta_X = -\frac{m^2P}{6EI}[3(X^2+X)(\tfrac{1}{2}+e)+X^3-X]$$

which, on solving for the boundary conditions at the right-hand end, gives

$$v_X^s = \frac{mP}{6K_s}(N^3-X^3-N+X)$$

$$\theta_X = -\frac{m^2P}{24EI}[X^4-N^4-X^2+N^2+4e(X^3-N^3-X+N)]. \qquad (27)$$

Using the above expression to determine θ_{X+1} in the second of eqns (24) and solving for zero displacement at the right-hand end gives

$$v_X^m = \frac{m^3P}{120EI}[X^5-X(5N^4-5N^2+1)+4N^5-5N^3+N$$
$$+20e(N^4-N^3X+NX-N^2)+20e^2(N^3-X^3+X-N)]. \qquad (28)$$

The total displacement v_X is the sum of $v_X{}^s$ and $v_X{}^m$. For most common problems, e is zero, which simplifies these expressions considerably.

In the case shown at the top of Fig.4 when no distributed load is applied,

$$\theta_0 = M_0\frac{mN}{EI} - S_0\frac{m^2N}{EI}(e+\tfrac{1}{2}N)$$

$$v_0 = M_0\frac{m^2N}{EI}(e+\tfrac{1}{2}N) - S_0\frac{m^3N}{12EI}[4N^2-1+12e(e+N)] - S_0\frac{mN}{K_s}. \qquad (29)$$

By similar methods to those described earlier, these expressions can be developed to give the slope–deflection equations which could be used when modular beams form the components of macro-structures. These take the form

$$M_0 = \frac{EI}{m^2N(N^2+s)}\{m\theta_0[4N^2+s+12e(e+N)]+m\theta_N[2N^2-s-12e^2]-6\delta(N+2e)\}$$

$$-M_N = \frac{EI}{m^2N(N^2+s)}\{m\theta_0[2N^2-s-12e^2]+m\theta_N[4N^2+s+12e(e-N)]-6\delta(N-2e)\}$$

$$S_0 = S_N = \frac{6EI}{m^3N(N^2+s)}[m\theta_0(N+2e)+m\theta_N(N-2e)-2\delta] \qquad (30)$$

where

$$s = \frac{12EI}{m^2 K_s} - 1. \tag{31}$$

If m is allowed to tend to zero in the above equations, mN remaining finite, they become the slope–deflection equations for a prismatic beam.

Other standard solutions are as follows. For a uniformly-loaded cantilever with a force P at each module junction, but only $\frac{1}{2}P$ at the tip, the tip deflection is

$$v_0 = \frac{Pm^3}{8EI}[N^4 + 4eN^2(e+N)] + \frac{PmN^2}{2K_s} \tag{32}$$

where e is the offset towards the fixed end, as in Fig.4. For a simply-supported beam with $2N$ modules, with a downwards force F applied at the centre, the central deflection is

$$v_N = \frac{Fm^3}{24EI}[4N^3 - N + 12eN(e+N)] + \frac{FmN}{2K_s}. \tag{33}$$

Here, e is the offset towards the centre and one of the simple supports is on rollers. (The assumption that there is no resultant axial force in the modular beam is implicit in the analysis.) For the same beam supporting a uniform distributed load P at each module junction,

$$v_N = \frac{Pm^3}{24EI}[5N^4 - 2N^2 + 12eN^2(e+N)] + \frac{PmN^2}{2K_s}. \tag{34}$$

If the ends of the above beam are fixed and a central point load F applied,

$$v_N = \frac{Fm^3}{24EI}(N^3 - N) + \frac{FmN}{2K_s}. \tag{35}$$

(Note the absence of terms in the offset e in this instance.) Finally, if instead this beam carries a uniform load P at each junction,

$$v_N = \frac{Pm^3}{24EI}[N^4 - N^2 - 2eN(N^2-1)] + \frac{PmN^2}{2K_s}. \tag{36}$$

If e is zero and $1/K_s$ is replaced by $(1/K_s - m^2/6EI)$ the expressions given by eqns (33), (35) and (36) would have been predicted by treating the modular beam as if it were an ordinary continuum. This approach has been used by Saka and Heki (1981). They derived shear stiffnesses for specific pin-jointed trusses which correspond to this substitution. However, eqns (32) and (34) show that it cannot be relied on as a general rule. Also, from energy considerations, one would expect this flexibility to be positive-definite.

6. EXAMPLES

The only modular structures considered here will be pin-jointed or rigid-jointed plane trusses. The analysis is applicable to any regular modular system, and the flexibilities of pin-jointed space trusses is the subject of another paper (Renton, 1995). Some of the basic

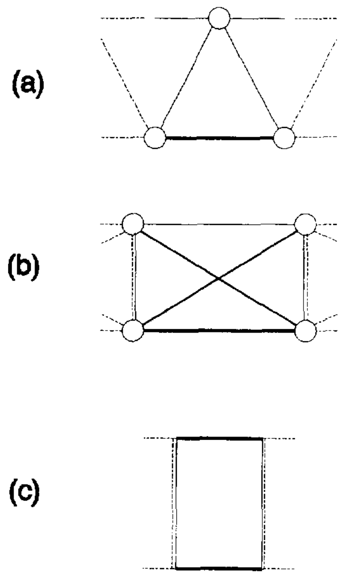

Fig. 5. Plane truss modules.

properties can be discussed using the three modules shown in Fig.5. The breadth and depth of each of the modules will be denoted by m and h, respectively. Their flexibilities are most readily expressed in terms of the axial-stiffness parameters EA/l^3 and the flexural-stiffness parameters EI/m^2h^2l of the component bars of the module, where l is the length of a component bar, A is its cross-sectional area, and the other terms have been defined previously. The axial-stiffness parameters for the top and bottom horizontal bars will be denoted by t and b, respectively, and those for any vertical or diagonal bars by v or d, respectively. The flexural-stiffness parameters for the horizontal and vertical bars in the third module (which is the only rigid-jointed unit) will be denoted by H and V, respectively.

The module flexibilities for the Warren truss shown in Fig.5(a) are then

$$\frac{1}{EI}=\frac{1}{m^3h^2}\left(\frac{1}{b}+\frac{1}{t}\right),\quad \frac{1}{K_s}=\frac{1}{mh^2}\left(\frac{2}{d}+\frac{1}{4t}\right). \tag{37}$$

If the module is shifted by half a bay-width either way, so that the lower chord bars are bisected instead of the upper ones, t is replaced by b in the expression for the shear flexibility. Note that the bending flexibilty could have been derived from an imaginary beam consisting of the upper and lower chord members only. This has been a common approximation used in estimating the deflections of trusses, but is inappropriate for the cross-braced truss in Fig.5(b). For this, the flexibilities are

$$\frac{1}{EI}=\frac{2}{m^3h^2}\left[\frac{2dv+(b+t)(2d+v)}{dv(b+t)+2bt(2d+v)}\right],\quad \frac{1}{K_s}=\frac{1}{mh^2}\left(\frac{1}{2d}+v\left[\frac{d(b-t)}{dv(b+t)+2bt(2d+v)}\right]^2\right). \tag{38}$$

If the top and bottom chord stiffnesses are the same, the bending flexibility again reduces to the previous simple approximation. The reason that it works in both cases is that pure flexure without axial loading occurs without any strains being induced in the bars between the upper and lower chords. In the former case, this is evident from statics, and in the latter case, the intermediate bars form mechanisms permitting the relative movements of the upper and lower joints to be in equal and opposite senses. The module of a Vierendeel truss shown in Fig. 5(c) will be taken to have identical upper and lower chord members, so that b and t are equal. Its flexibilities are then

$$\frac{1}{EI}=\frac{2}{m^3h^2(4H+b)},\quad \frac{1}{K_s}=\frac{1}{6mh^2(4H+b)}\left[1+\frac{b}{4H}+\frac{b^2}{2V(4H+b)}\right]. \tag{39}$$

In all three cases, the eccentricity e of the flexural centre from the middle of the module is zero.

The module of the Warren truss is statically determinate, so that the characteristic mode of response to a resultant bending moment and shear force is in fact the only possible response. Also, the lower joints of the truss are the only ones on the module boundaries. This means that the resultant shear forces acting on the modules and the deflections through which they do work correspond to actual joint loads and displacements. It follows that the above analyses give exact results for the displacements of the lower joints of such trusses, if these are the joints which are loaded and supported. The fixed-ended solutions given by eqns (35) and (36) are also exact if the upper and lower chord members have the same axial stiffnesses. This is because no resultant axial forces are then induced by the supports (it being implicit in the analysis that the modular beam is subject to resultant bending moments and shear forces only). Suppose for example that the bars of the truss are all of the same unit length and axial stiffness. The EI value for the module is then 3/8 and the K_s value is 1/3. If the truss is a cantilever consisting of five modules with a unit end shear force applied, the tip displacement can be deduced from the second of eqns (29) and is 125 units, 110 units resulting from the flexural terms and 15 units from the shear term. This total is exact (as can be checked by computer analysis). The shear deflection is then 13.6% of the flexural deflection. For a rectangular beam in plane stress of the kind described earlier, but with the same depth/span ratio as the Warren truss, the shear deflection is only 2.32% of the flexural deflection. This indicates that the shear behaviour of trusses is more significant than that of ordinary beams. It will be seen later that it can be the dominant effect.

The module of the cross-braced truss shown in Fig.5(b) is not statically determinate, and its boundary loads and displacements are not associated with single joints. For this reason, the results are no longer exact. As an example, the EA value for the top chord bars will be taken as one unit, and taken as six units for the other bars. The values of m and h will be taken as two and one units, respectively. This gives an EI value for the module of 1.312 and a K_s value of 2.060. The EI value, given by the simple analogy with a beam consisting of the upper and lower chords only, is 0.8571. The tip deflection of this truss acting as a cantilever with an end shear force can be analysed as before. Comparisons between the results predicted by different methods are made in the following table.

Tip deflections of cross-braced trusses

N	Computer	Simple analogy	% error	Modular beam	% error
1	2.424	3.111	28.5	2.495	2.92
2	16.87	24.89	47.5	17.18	1.84
3	55.53	84.00	51.3	56.25	1.29
4	130.6	199.1	52.5	131.9	0.98
5	254.2	388.9	53.0	256.3	0.81

The Vierendeel truss becomes a multi-storey portal frame when turned through 90°. Lin and Stotesbury (1981), in discussing the design of skyscrapers, comment on the relative importance of overall bending and racking (shear). They state that 'The cantilever deflection due to column shortening and lengthening (produced by overall turning moment) is usually of secondary importance until the building is some 40 stories or higher'. Consider the case where such a frame is subject to a uniform side pressure p, so that eqn (32) applies. In Fig. 6, p is taken as 200 N/m, the height of each storey (m) as 5 m and its width (h) as 10 m and the beams used have a common section with a cross-sectional area of 100 cm^2, second moment of area 40,000 cm^2 and Young's modulus of 200 GPa. The deflected forms, generated by normal computer structural analysis, are much exaggerated of course. The circles indicate points of *contracurvature*. As the frame is subject to a distributed side load, the resultant shear force, and hence the shear slope, diminishes upwards from the base.

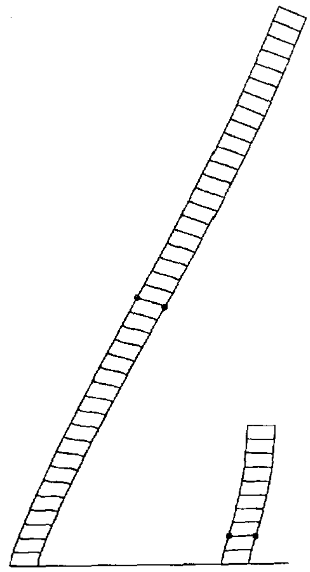

Fig. 6. Deflections of multi-storey portals.

However, the flexural slope is zero at the base and increases with height. At the points of contracurvature, flexure-dominant behaviour gives way to shear-dominant behaviour. (The term contraflexure would obviously be inappropriate under these circumstances.) The ten-storey frame on the right is thus mostly dominated by shear effects, but the forty-storey frame on the left is about equally dominated by each. The module bending stiffness EI is 100.2 GN m^2 and its shear stiffness K_s is 15.40 MN. The computer results and the results obtained from eqn (32) are compared in the following table. (L is understood to be the overall length, that is, mN.)

Tip deflections of multi-storey portal frames

Stories	Computer deflection (mm)	Bending deflection $pL^4/8EI$	Shear deflection $pL^2/2K_s$	Bending + shear
10	15.91	1.56	16.23	17.79
20	86.03	24.96	64.94	89.90
30	267.15	126.36	146.11	272.47
40	653.75	399.36	259.75	659.11

In this case, the contribution of flexure to the sideways displacement at the top becomes dominant when the number of stories is greater than 32.

7. CONCLUSIONS

In establishing the basis of the engineering theory of beams, it has proved necessary to discard several commonly-held beliefs about its nature. Although originally based on the hypothesis that plane sections remain plane, this is not a fundamental assumption, but emerges as the correct condition for the bending of homogenous, isotropic beams. Attempts to generalise this hypothesis, such as that of Noor *et al.* (1978), can be misleading. Likewise, the appropriate beam deflections to use may not be those of the centroidal axis, as assumed by Timoshenko and Goodier (1970) or Donnell (1976). Instead, the displacement of a beam section is defined as that through which the resultant shear force does work, and its rotation as that through which the resultant moment does work. This has implications, not only for the formulation of the governing equations but also for understanding end conditions. Equations (26) show that there is a relationship between the derivatives of the flexural displacement, the shear displacement and the section rotation. However, integrating these relationships and discarding the coefficients of integration as proposed by Donnell [*ibid.* (3.60)], may not be justified on the grounds suggested. This means that some of the simple geometrical relationships between these parameters may be lost. The use of energy methods seems to be both more general in application and less vulnerable to implicit assumptions than geometrical methods. The investigation also illustrates that beam theory is only concerned with macroscopic behaviour. That is, the overall rotation and displacement of a beam in response to resultant moments and shear forces. No more can be said about the finer details unless localised behaviour is analysed in terms of stress and strain patterns which decay along the beam.

Previously, beam theory was applied to give the deflections of trusses by trying to establish the properties of equivalent continua. As the shear behaviour of trusses is usually more significant than the shear behaviour of beams, it is important to include this in the analysis. However, if the continuum analogy is used, the shear stiffness of the truss appears to vary with the loading and end conditions. It has been seriously suggested that it is a function of these conditions and not an innate property of the truss. The present analysis shows that this apparent variability results from a shortcoming of the continuum analogy. The direct application of generalized beam theory to finite-difference analysis uses stiffnesses which are constant for any truss and produces additional shear-like terms associated with the flexural stiffness, accounting for the above apparent shear-stiffness variability. The internal stresses can be determined from the characteristic responses to the resultant loads. Using the finite-difference approach, engineering beam theory may be as accurate as when it is applied to ordinary prismatic beams. Indeed, for statically-determinate trusses it can be exact.

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APPENDIX

The summation formula for polynomials is

$$\sum_{X=1}^{N} X^p = \frac{1}{p+1}[(N-B)^{p+1} - (-B)^{p+1}]$$

where B^n in the binomial expansion of the right-hand side is understood to be the nth Bernoulli number. The solution of the finite-difference equations is aided by use of the following table.

$f(X)$	$\Delta^{-1}f(X)$	$(\Delta\nabla)^{-1}f(X)$
1	X	$\frac{1}{2}X^2$
X	$\frac{1}{2}X(X-1)$	$\frac{1}{6}X^3$
X^2	$\frac{1}{6}X(X-1)(2X-1)$	$\frac{1}{12}(X^4-X^2)$
X^3	$[\frac{1}{2}X(X-1)]^2$	$\frac{1}{20}X^5-\frac{1}{12}X^3$
X^4	$\frac{1}{5}X^5-\frac{1}{2}X^4+\frac{1}{3}X^3-\frac{1}{30}X$	$\frac{1}{30}X^6-\frac{1}{12}X^4+\frac{1}{20}X^2$

where a constant must be added to the terms in the second column and and an arbitrary linear expression in X added to the terms in the third column. Further formulae will be found in Jordan (1965) for example.